

A NOTE ON UNIFORM ORDERED SPACES

HANS-PETER A. KÜNZI

(Received 15 July 1985; revised 3 January 1986)

Communicated by J. H. Rubinstein

Abstract

We characterize the generalized ordered topological spaces X for which the uniformity $\mathcal{U}(X)$ is convex. Moreover, we show that a uniform ordered space for which every compatible convex uniformity is totally bounded, need not be pseudocompact.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 06 B 30, 54 F 05.

We answer two questions of P. Fletcher and W. F. Lindgren. In [2, Problem F, page 94] they ask for necessary and sufficient conditions such that the uniformity $\mathcal{U}(X)$ of a *GO* space (X, \mathcal{T}, \leq) is convex with respect to \leq . In this note we show that the uniformity $\mathcal{U}(X)$ of a *GO* spaces X is convex if and only if each closed discrete subset of X is countable. In [2, Problem I, p. 95] they ask whether a uniform ordered space (X, \mathcal{T}, \leq) for which every convex uniformity compatible with \mathcal{T} is totally bounded, is necessarily pseudocompact. They observe that the answer is positive, if \leq is a linear order on X . In this note we show that the answer is negative in general. We will use the notation and the terminology of [2].

1.

In the first part of this note we will need the following lemma.

LEMMA. *Let (X, \leq) be an uncountable linearly ordered set. Then there exists an uncountable subset A of X such that, if $a, b \in A$ and $a \leq b$, then there is a $c \in X \setminus A$ with $a \leq c \leq b$.*

PROOF. Denote by $[X]^3$ the set of the subsets of X with three elements. If $B \in [X]^3$, we denote the minimal element of B by L_B and the maximal element of B by R_B . Finally, M_B denotes the element of B such that $L_B < M_B < R_B$.

Case 1. There exists a countable subset D of X so that for each $B \in [X]^3$ there is a $d \in D$ such that $L_B \leq d \leq R_B$. Set $C = X \setminus D$ and $A = \{x \in X \mid x \text{ is the smallest element of a convexity-component of } C \text{ in } X\}$. Since each convexity-component of C in X has at most two elements, A is an uncountable set that satisfies the condition of the lemma.

Case 2. For each countable subset D of X there is a $B \in [X]^3$ such that $D \cap \{x \in X \mid L_B \leq x \leq R_B\} = \emptyset$. Define by transfinite induction for each $\beta < \omega_1$ a set $B(\beta) \in [X]^3$: Suppose that $B(\alpha)$ has been defined for each $\alpha < \beta$. There is a set $B \in [X]^3$ such that $\{x \in X \mid L_B \leq x \leq R_B\}$ contains no element of the countable set $\bigcup\{B(\alpha) \mid \alpha < \beta\}$. Set $B(\beta) = B$. Then $A = \{M_{B(\beta)} \mid \beta < \omega_1\}$ is uncountable and satisfies the condition of the lemma.

It is known that each GO space is normal. In the next proof we will use the result that for a normal T_2 -space X the uniformity $\mathcal{U}(X)$ is the finest compatible uniformity on X if and only if each locally finite open cover of X has a countable open refinement of finite order [3, Remark following the proof of the theorem; compare 1 and 2, p. 190, §5.28]. Recall that a topological space X is called ω_1 -compact, if each closed discrete subset of X is countable.

PROPOSITION. *Let (X, \mathcal{T}, \leq) be a GO space. Then the following conditions are equivalent:*

- (a) X is ω_1 -compact.
- (b) $\mathcal{U}(X)$ is the finest uniformity for (X, \mathcal{T}) .
- (c) $\mathcal{U}(X)$ is convex.

PROOF. (a) \rightarrow (b). Since X is ω_1 -compact and $\dim X \leq 1$, every locally finite open cover of X is refined by a countable open refinement of finite order. We conclude that $\mathcal{U}(X)$ is the finest uniformity for (X, \mathcal{T}) .

(b) \rightarrow (c). Since the finest uniformity for a GO space is convex [2, Theorem 4.33], $\mathcal{U}(X)$ is convex.

(c) \rightarrow (a). Let $\mathcal{U}(X)$ be convex. Assume that X is not ω_1 -compact. Then X has an uncountable closed discrete subset A . By the lemma there exists an uncountable subset A' of A such that every subset of A' that is convex in A contains at most one point. Define a function $f: A \rightarrow R$ by $f(x) = 0$ if $x \in A'$, and $f(x) = 1$ if $x \in A \setminus A'$. Let $g: X \rightarrow R$ be continuous such that $g|_A = f$. Let

$V \in \mathcal{C}(X)$ such that $V \subset \{(x, y) \in X \times X: |g(x) - g(y)| < \frac{1}{2}\}$ and $V(x)$ is convex in X for each $x \in X$. Since $V \in \mathcal{C}(X)$, there is a countable subset D of X such that $X = \bigcup\{V(d) | d \in D\}$. Clearly, each $V(d)$ contains at most one element of A' —a contradiction. We conclude that X is ω_1 -compact.

EXAMPLE 1. Let R denote the set of the reals and let \leq denote the usual order on R . Consider the *GO* space (R, \mathcal{T}, \leq) where \mathcal{T} denotes the discrete topology on R . Clearly (R, \mathcal{T}) is not ω_1 -compact. Hence $\mathcal{C}(R)$ is not convex.

2.

We construct a uniform ordered space (X, \mathcal{T}, \leq) such that every convex uniformity compatible with (X, \mathcal{T}, \leq) is totally bounded, but X is not pseudocompact.

EXAMPLE 2. We use a modification of the well-known pseudocompact space ψ [see 4, 51]. Let N be the set of the positive integers and let Γ be an infinite maximal almost disjoint family of infinite subsets of N . As usual set $\psi = N \cup \Gamma$. Let $(a_{2n-1})_{n \in N}$ be a sequence of pairwise different elements of Γ . Set $a_{2n} = -n$ for each $n \in N$, $A = \{a_{2n} | n \in N\}$, and $D = \{a_n | n \in N\}$. Let $X = N \cup \Gamma \cup A$. Consider the following collection of subsets of X :

$$\mathcal{B} = \{\{n\} | n \in N\} \cup \{(E \setminus F) \cup \{E\} | E \in \Gamma \setminus D, F \text{ is a finite subset of } N\} \cup \{\bigcup_{n=1}^k G(a_n) | k \in N; G(a_{2m}) = \{a_{2m}\} \text{ (for each } m \in N \text{ such that } 2m \leq k); G(a_{2m-1}) = (a_{2m-1} \setminus F) \cup \{a_{2m-1}\} \text{ where } F \text{ is a finite subset of } N \text{ (for each } m \in N \text{ such that } 2m - 1 \leq k)\}\}.$$

Set $\mathcal{S} = \{[X \times G] \cup [(X \setminus G) \times X] | G \in \mathcal{B}\}$. Then \mathcal{S} is a subbase for a quasi-uniformity \mathcal{U} on X . Consider the topology $\mathcal{T}(\mathcal{U}^*)$ on X where \mathcal{U}^* denotes the uniformity generated by $\{V \cap V^{-1} | V \in \mathcal{U}\}$ on X . One easily checks that the points of the $\mathcal{T}(\mathcal{U}^*)$ -open subspace ψ of X have their usual neighbourhoods. Moreover, each point of $X \setminus \psi$ is isolated in X . Hence $\mathcal{T}(\mathcal{U}^*)$ is not pseudocompact. Since $\mathcal{T}(\mathcal{U}^*)$ is a Hausdorff topology, $(X, \mathcal{T}(\mathcal{U}^*), \cap \mathcal{U})$ is a uniform ordered space [2, pages 81, 84]. Let \mathcal{W} be a convex uniformity compatible with $(X, \mathcal{T}(\mathcal{U}^*), \cap \mathcal{U})$. Let $Z \in \mathcal{W}$. Since \mathcal{W} is convex, there is a $Z' \in \mathcal{W}$ such that $Z' \subset Z$ and $Z'(x)$ is convex in X for each $x \in X$. Since ψ is a pseudocompact subspace of X , there is a finite subset F of ψ such that $\psi \subset Z'(F)$. Hence there is an $x \in F$ such that $Z'(x)$ contains infinitely many points of $D \setminus A$. Note that, if

$k, n \in N$ and $k \leq n$, then $(a_n, a_k) \in \cap \mathcal{U}$. Since $Z'(x)$ is convex in X , $Z'(x)$ contains all but finitely many points of D . We conclude that \mathcal{W} is totally bounded.

References

- [1] H. H. Corson and J. R. Isbell, 'Euclidean covers of topological spaces', *Quart. J. Math. Oxford Ser. (2)* **11** (1960), 34–42.
- [2] P. Fletcher and W. F. Lindgren, *Quasi-uniform spaces* (Lecture Notes in Math., Dekker, New York, 1982).
- [3] P. Fletcher, H. Junnila and W. Lindgren, 'Another note on bounded paracompactness', manuscript.
- [4] L. Gillman and M. Jerison, *Rings of continuous functions* (Van Nostrand, Princeton, N. J., 1960).

Department of Mathematics
University of Bern
Sidlerstr. 5, 3012 Bern
Switzerland